JOURNAL OF NUMBER THEORY 11, 14–15 (1979)

On the Partial Quotients of Algebraic Integers

RALPH P. PASS

NASA/Goddard Space Flight Center, Operational Orbit Support Branch Code 572, Greenbelt, Maryland 20771

Communicated by S. Chowla

Received October 28, 1977; revised April 28, 1978

It is shown that there are algebraic integers, with degree greater than 2, having infinitely many large partial quotients in their simple continued fraction expansions. This generalizes an earlier result of Davenport for algebraic numbers.

Properties of the partial quotients of the continued fraction expansions of algebraic numbers having degree greater than 2 have been elusive. Roth's theorem [3] provides a liberal limit on the growth rate of the partial quotients of algebraic numbers and Davenport [1] has shown that there are algebraic numbers with infinitely many large partial quotients. This paper's purpose is to generalize Davenport's result to algebraic integers.

The results will be stated in terms of Diophantine approximation properties. The corresponding results on partial quotients of the continued fraction expansion are given in:

(1) If $|q\alpha - p| < 1/Nq$ with $N \geqslant 2$, then p/q is a principle convergent of α,

(2) If $|q\alpha - p| < 1/Nq$ and $p/q = p_n/q_n$ is a principle convergent, then $a_{n+1} > N - 2$ [see [2], pp. 1-11 for notation and results].

THEOREM 1. *Given $N > 0$, there is an algebraic integer β of degree > 2, such that:*

$$|q\beta - p| < 1/Nq \tag{1}$$

has infinitely many integer pair solutions (p, q).

Our result will depend on Davenport's theorem [1]:

THEOREM 2. *Given a prime p and an algebraic number α then β, one of $p\alpha$ or $(\alpha + c)/p$ for some c in $\{0, \ldots, p-1\}$, satisfies* (1) *for infinitely many integer pairs (p, q).*

0022-314X/79/010014–02$02.00/0

Proof of Theorem 1. For integers $n \geqslant 1$, define $g_n(x) = x(x+1) \cdots (x+n-1)$. Let p be a prime, with $p \geqslant N+2$, and let $f(x) = pg_p(x) - 1$. By Eisenstein's criteria applied to $x^p f(1/x)$, we see that f is irreducible, and since $f(0) < 0, f(1) > 0$; f has a real root, say α. Apply Theorem 2 to α and p, then there is a β satisfying (1) with $N = p$ for infinitely many pairs (p, q). If $\beta = p\alpha$, then β is an algebraic integer and we are done. Otherwise, suppose $\beta = (\alpha + c)/p$ for some integer $c \in \{0,..., p-1\}$. The minimal polynomial of β is $h(x) = f(px - c)$ which by construction has the form:

$$h(x) = xk(x) - 1$$

for some $k(x) \in Z[x]$.

Thus, $1/\beta$ is an algebraic integer and if β has infinitely many solutions to (1), then so does $1/\beta$, proving the result.

Actually, this is a weak generalization of Davenport's result since his β had the same degree as his α, which was arbitrary. In Theorem 1, the degree increases with N, and hence doesn't generalize Davenport's result to algebraic integers of arbitrary degrees.

References

1. H. Davenport, A remark on continued fractions, *Michigan Math. J.* **11** (1964), 343–344.
2. S. Lang, "Introduction to Diophantine Approximations," Addison–Wesley, Reading, Mass., 1966.
3. K. F. Roth, Rational approximations to algebraic numbers, *Mathematika* **2** (1955), 1–20.